

ON SIMPLE POLYNOMIAL  $G_rT$ -MODULES

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ABSTRACT. Using the general framework of polynomial representations defined by Doty and generalizing the definition given by Doty, Nakano and Peters for  $G = \mathrm{GL}_n$ , we consider polynomial representations of  $G_rT$  for an arbitrary closed reductive subgroup scheme  $G \subseteq \mathrm{GL}_n$  and a maximal torus  $T$  of  $G$  in positive characteristic. We give sufficient conditions on  $G$  making a classification of simple polynomial  $G_rT$ -modules similar to the case  $G = \mathrm{GL}_n$  possible and apply this to recover the corresponding result for  $\mathrm{GL}_n$  with a different proof, extending it to symplectic similitude groups, Levi subgroups of  $\mathrm{GL}_n$  and, in a weaker form, to odd orthogonal similitude groups. We also consider orbits of the affine Weyl group and give a condition for equivalence of blocks of polynomial representations for  $G_rT$  in the case  $G = \mathrm{GL}_n$ .

## 1. INTRODUCTION

When considering rational representations of  $G = \mathrm{GL}_n$  over an infinite field, one can reduce many questions to polynomial representations. These representations correspond to representations of certain finite-dimensional algebras, the so-called Schur algebras (see for example [6, 8]). Letting  $T$  be a maximal torus of  $G$  and  $k$  be an infinite field of positive characteristic, one can also study the category of  $G_rT$ -modules to get information on the representation theory of  $G$  (see [7, II.9]). In [4], Doty, Nakano and Peters combined these approaches and considered polynomial representations of  $G_rT$  for  $G = \mathrm{GL}_n$ . They show that these representations correspond to representations of the so-called infinitesimal Schur algebras which are subalgebras of the ordinary Schur algebras. In [1], Doty defined a general notion of polynomial representations and analogues of Schur algebras for algebraic groups contained in  $\mathrm{GL}_n$ . Using his definition, one can study polynomial representations of  $G_rT$  for other algebraic groups  $G$ . If a reductive group  $G \subseteq \mathrm{GL}_n$  admits a graded polynomial representation theory in the sense of [1], it is natural to ask for which character  $\lambda$  contained in the character group  $X(T)$  of  $T$  the simple  $G_rT$ -module  $\widehat{L}_r(\lambda)$  is a polynomial  $G_rT$ -module. This problem was solved for  $G = \mathrm{GL}_n$  in [4], but in contrast to most results of that paper, the proof does not carry over to other reductive groups. Let  $D = \overline{T}$  be the closure of  $T$  in the algebraic monoid of  $(n \times n)$ -matrices,  $P(D) \subseteq X(T)$  the set of polynomial weights, i.e. the character monoid of  $D$ , and  $X_r(T) \subseteq X(T)$  the

set of  $p^r$ -restricted weights. Then our main result can be summarized as follows:

- Theorem.** (a) Let  $G = GSp_{2l}$  be the symplectic dilation group,  $T \subseteq G$  the torus of diagonal matrices. Then a complete set of representatives for the isomorphism classes of simple polynomial  $G_r T$ -modules is given by  $\{\widehat{L}_r(\lambda) \mid \lambda \in P_r(D) + p^r P(D)\}$ , where  $P_r(D) = \{\lambda \in P(D) \cap X_r(T) \mid \lambda - p^r d \notin P(D)\}$  and  $d$  is the character of  $T$  given by  $t \mapsto t_{11} t_{nn}$ .
- (b) Let  $G = GO_{2l+1}$  be the connected component of the odd orthogonal dilation group,  $T$  the torus of diagonal matrices in  $G$ . If  $\lambda \in X_r(T) + p^r X(T)$  and all weights of the module  $\widehat{L}_r(\lambda)$  are polynomial, then  $\lambda \in P_r(D) + p^r P(D)$ , where  $P_r(D) = \{\lambda \in P(D) \cap X_r(T) \mid \lambda - p^r d \notin P(D)\}$  and  $d$  is the character of  $T$  given by  $t \mapsto t_{l+1}$ .
- (c) Let  $n_i \in \mathbb{N}$  such that  $n = \sum_{i=1}^l n_i$  and embed  $G = GL_{n_1} \times \dots \times GL_{n_l}$  into  $GL_n$  as a Levi subgroup in the obvious way. Let  $T$  be the torus of diagonal matrices in  $G$ ,  $M_i = \{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$  and  $d_i = \sum_{j \in M_i} e_j$  for  $1 \leq i \leq l$ , where  $e_1, \dots, e_n$  is the standard basis of  $X(T) \cong \mathbb{Z}^n$ . Then a complete set of representatives for the isomorphism classes of simple polynomial  $G_r T$ -modules is given by  $\{\widehat{L}_r(\lambda) \mid \lambda \in P_r(D) + p^r P(D)\}$ , where  $P_r(D) = \{\lambda \in X_r(T) \cap P(D) \mid \lambda - p^r d_i \notin P(D) \text{ for } 1 \leq i \leq l\}$ .

We derive this as a corollary of a more general and technical result giving a sufficient condition on  $G$  for a classification of simple polynomial  $G_r T$ -modules. Our result also gives a new proof for the case  $G = GL_n$  originally treated in [4] and allows all cases to be treated in a uniform way. By studying the intersection of orbits of the affine Weyl group with suitable sets of polynomial weights, we also show that certain shift functors induce Morita equivalences between blocks of polynomial representations of  $G_r T$  for  $G = GL_n$ .

## 2. NOTATION AND PREREQUISITES

In this section, we fix notation and recall some basic results on polynomial representations. For algebraic groups and  $G_r T$ , we follow the notation from [7]. For an introduction to algebraic monoids, we refer the reader to [10].

Let  $k$  be an infinite perfect field of characteristic  $p > 0$  and  $\text{Mat}_n$  be the monoid scheme of  $(n \times n)$ -matrices over  $k$ . Let  $d \in \mathbb{N}_0$  and denote by  $A(n, d)$  the space generated by all homogeneous polynomials of degree  $d$  in the coordinate ring  $k[\text{Mat}_n] = k[X_{ij} \mid 1 \leq i, j \leq n]$  of  $\text{Mat}_n$ . Then  $k[\text{Mat}_n] = \bigoplus_{d \geq 0} A(n, d)$  is a graded  $k$ -bialgebra with comultiplication  $\Delta : k[\text{Mat}_n] \rightarrow k[\text{Mat}_n] \otimes_k k[\text{Mat}_n]$  and counit  $\epsilon : k[\text{Mat}_n] \rightarrow k$  given by  $\Delta(X_{ij}) = \sum_{l=1}^n X_{il} \otimes_k X_{lj}$ ,  $\epsilon(X_{ij}) = \delta_{ij}$  and each  $A(n, d)$  is a subcoalgebra. As  $GL_n$  is dense in  $\text{Mat}_n$ , the canonical map  $k[\text{Mat}_n] \rightarrow k[GL_n]$  is

injective, so that  $k[\text{Mat}_n]$  and each  $A(n, d)$  can be viewed as a subcoalgebra of  $k[\text{GL}_n]$ . Now let  $G$  be a closed subgroup scheme of  $\text{GL}_n$  and  $\pi : k[\text{GL}_n] \rightarrow k[G]$  the canonical projection. Set  $A(G) = \pi(k[\text{Mat}_n])$  and  $A_d(G) = \pi(A(n, d))$  as well as  $S_d(G) = A_d(G)^*$ . Since  $\pi$  is a homomorphism of Hopf algebras,  $A(G)$  is a subbialgebra of  $k[G]$  and each  $A_d(G)$  is a subcoalgebra of  $A(G)$ . As  $A_d(G)$  is finite dimensional,  $S_d(G)$  is a finite dimensional associative algebra in a natural way. For  $G = \text{GL}_n$  and  $T$  a maximal torus of  $G$ , the algebra  $S_d(G)$  is the ordinary Schur algebra, while  $S_d(G_rT)$  is the infinitesimal Schur algebra defined in [4]. Following [1], we say that  $G$  admits a graded polynomial representation theory if the sum  $\sum_{d \geq 0} A_d(G)$  is direct and we say that a rational  $G$ -module  $V$  is a polynomial  $G$ -module if the corresponding comodule map  $V \rightarrow V \otimes k[G]$  factors through  $V \otimes A(G)$ . If the comodule map factors through  $V \otimes A_d(G)$  for some  $d \in \mathbb{N}_0$ , we say that  $V$  is homogeneous of degree  $d$ . Clearly, every  $A_d(G)$ -comodule is an  $A(G)$ -comodule and every  $A(G)$ -comodule is a  $G$ -module in a natural way. We record some other properties of polynomial representations from [1, 1.2] in the following proposition.

**Proposition 2.1.** (1) *Suppose  $G$  admits a graded polynomial representation theory and  $V$  is a polynomial  $G$ -module. Then  $V = \bigoplus_{d \geq 0} V_d$ , where each  $V_d$  is homogeneous of degree  $d$ . Furthermore, the category of homogeneous modules of degree  $d$  is equivalent to the category of  $S_d(G)$ -modules.*  
 (2) *If  $G$  contains the center  $Z(\text{GL}_n)$  of  $\text{GL}_n$ , then  $G$  admits a graded polynomial representation theory.*

As  $A(G)$  is a factor bialgebra of  $k[\text{Mat}_n]$ , it corresponds to a closed submonoid  $M$  of  $\text{Mat}_n$ . Since  $A(G) \subseteq k[G]$  is a subbialgebra,  $G \subseteq M$  is a dense subscheme, so that  $M$  is the closure of  $G$  in  $\text{Mat}_n$ . Hence polynomial representations of  $G$  can be regarded as rational representations of the algebraic monoid scheme  $M = \overline{G}$ .

If  $G = \text{GL}_n$ , a comparison of coordinate rings shows  $\overline{G_rT} = M_rD$ , where  $M_rD = (F^r)^{-1}(D)$  with  $F^r$  the  $r$ -th iteration of the Frobenius homomorphism, see [4]. This can fail for general  $G$ : If  $G$  is the full orthogonal similitude group in dimension two and  $T$  the torus of diagonal matrices in  $G$ , then  $G_rT = T$ , but  $M_rD$  is strictly larger than  $D = \overline{T}$ . However, we always have  $\overline{G_rT} \subseteq M_rD$ . We do not know whether  $M_rD = \overline{G_rT}$  e.g. for connected  $G$ .

### 3. SIMPLE POLYNOMIAL $G_rT$ -MODULES

In this section, let  $G \leq \text{GL}_n$  be a closed connected reductive subgroup scheme containing  $Z(\text{GL}_n)$ ,  $T \subseteq G$  be a maximal torus contained in a maximal torus  $T' \subseteq \text{GL}_n$ ,  $M$  be the closure of  $G$  in  $\text{Mat}_n$ ,  $D$  be the closure of  $T$ ,  $D'$  be the closure of  $T'$  and  $P(D) \subseteq X(T)$  be the character monoid of  $D$ , i.e. the set of polynomial weights,  $W$  the Weyl group of

$G$ . Let  $R$  be the root system of  $G$  with respect to  $T$  and  $S \subseteq R$  a set of simple roots. Denote by  $\langle -, - \rangle : X(T) \times Y(T) \rightarrow \mathbb{Z}$  the canonical perfect pairing of  $X(T)$  and the cocharacter group  $Y(T)$ . For  $\alpha \in R$ , let  $\alpha^\vee$  be the coroot of  $\alpha$ . Let  $X_0(T) = \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in R\}$  as well as  $X_r(T) = \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p^r - 1 \text{ for all } \alpha \in S\}$ .

We want to determine the simple polynomial  $G_r T$ -modules. Recall that the isomorphism classes of simple  $G_r T$ -modules are parametrized by  $X(T)$  via  $\lambda \mapsto \widehat{L}_r(\lambda)$ , where  $\widehat{L}_r(\lambda)$  has highest weight  $\lambda$ , see [7, II.9]. As a motivation for our approach, consider the case where  $G = \mathrm{GL}_n$  and  $T \leq G$  is the maximal torus of diagonal matrices. According to [4, Section 3], the simple  $G_r T$ -module  $\widehat{L}_r(\lambda)$  is a polynomial  $G_r T$ -module iff  $\lambda \in P_r(D) + p^r P(D)$ , where  $P_r(D) = \{\lambda \in P(D) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq p^r - 1 \text{ for } 1 \leq i \leq n-1, 0 \leq \lambda_n \leq p^r - 1\}$ . A character  $\lambda \in X(T)$  belongs to  $P(D)$  iff  $\lambda_i \geq 0$  for  $1 \leq i \leq n$ , that is, iff  $\min\{\lambda_i \mid 1 \leq i \leq n\} \in \mathbb{N}_0$ . Taking this minimum is compatible with multiplication by natural numbers, in particular with multiplication by  $p^r$ . Given characters  $\lambda, \mu \in X(T)$ , we can of course find a permutation  $w \in W \cong S_n$  such that this minimum is attained at the same coordinate for  $w.\lambda$  and  $\mu$ , so that  $\min\{(w.\lambda + \mu)_i \mid 1 \leq i \leq n\} = \min\{\lambda_i \mid 1 \leq i \leq n\} + \min\{\mu_i \mid 1 \leq i \leq n\}$ . Writing  $d = \det|_T = (1, \dots, 1)$ , we have  $X_0(T) = \langle d \rangle$ ,  $\min\{d_i \mid 1 \leq i \leq n\} = 1$  and we can write  $P_r(D) = \{\lambda \in X_r(T) \cap P(D) \mid \lambda - p^r d \notin P(D)\}$ . We axiomatize these properties of  $\det|_T$  and the function  $X(T) \rightarrow \mathbb{Z}, \lambda \mapsto \min\{\lambda_i \mid 1 \leq i \leq n\}$  for general  $G$  in the following

**Assumption 3.1.** *Suppose there exists a function  $\varphi : X(T) \rightarrow \mathbb{Z}^l$  with the following properties:*

- (1)  $\forall \lambda \in X(T) : \varphi(\lambda) \in \mathbb{N}_0^l \iff \lambda \in P(D)$ ,
- (2)  $\forall \lambda \in X(T) : \varphi(p^r \lambda) = p^r \varphi(\lambda)$ ,
- (3)  $\forall \lambda, \lambda' \in X(T) : \exists w \in W : \varphi(w.\lambda + \lambda') = \varphi(\lambda) + \varphi(\lambda')$ ,
- (4)  $\varphi|_{X_0(T)} : X_0(T) \rightarrow \mathbb{Z}^l$  is bijective.

Since  $W$  acts trivially on  $X_0(T)$ , (1), (3) and (4) imply that the restriction to  $X_0(T)$  of any function  $\varphi$  as in 3.1 is an isomorphism mapping  $P(D) \cap X_0(T)$  to  $\mathbb{N}_0^l$ . It will turn out that 3.1 is sufficient for a classification similar to the case  $G = \mathrm{GL}_n$ . We first show that such a function is essentially unique.

**Proposition 3.2.** *If  $\varphi, \psi : X(T) \rightarrow \mathbb{Z}^l$  are functions as in 3.1, there is a permutation of coordinates  $\sigma : \mathbb{Z}^l \rightarrow \mathbb{Z}^l$  such that  $\varphi = \sigma \circ \psi$ .*

*Proof.* Let  $e_1, \dots, e_l$  be the standard basis of  $\mathbb{Z}^l$  and  $d_1, \dots, d_l$  resp.  $d'_1, \dots, d'_l$  be the preimages of the  $e_i$  in  $X_0(T)$  for  $\varphi$  resp.  $\psi$ . Then every element of  $X_0(T) \cap P(D)$  can be written as a linear combination in the  $d_i$  with non-negative coefficients. Writing the  $d'_i$  as such a

linear combination and then writing the  $d_i$  as such a linear combination in the  $d'_i$ , we see that the base change matrix for the two bases is a permutation matrix. Thus, there is  $\sigma \in S_l$  such that  $d_i = d'_{\sigma(i)}$  for  $1 \leq i \leq l$ . Now let  $\lambda \in X(T)$  and set  $\tilde{\lambda} = \lambda - \sum_{i=1}^l \varphi(\lambda)_i d_i$ . Using (3) and that  $W$  acts trivially on  $X_0(T)$  for the first equality and the fact that  $\psi|_{X_0(T)}$  is an isomorphism for the second equality, we get  $\psi(\tilde{\lambda}) = \psi(\lambda) + \psi(-\sum_{i=1}^l \varphi(\lambda)_i d_i) = \psi(\lambda) - \sum_{i=1}^l \varphi(\lambda)_i \psi(d_i) = \psi(\lambda) - \sum_{i=1}^l \varphi(\lambda)_i \psi(d'_{\sigma(i)}) = \psi(\lambda) - \sum_{i=1}^l \varphi(\lambda)_i e_{\sigma(i)}$ . Applying the same arguments to  $\varphi$ , we get  $\varphi(\tilde{\lambda}) = 0$  and  $\varphi(\tilde{\lambda} - d_i) = -e_i$  for  $1 \leq i \leq l$ . Thus,  $\tilde{\lambda} \in P(D)$  and  $\tilde{\lambda} - d_i = \tilde{\lambda} - d'_{\sigma(i)} \notin P(D)$  for  $1 \leq i \leq l$ , so that  $\psi(\tilde{\lambda}) \in \mathbb{N}_0^l$  and  $\psi(\tilde{\lambda}) - e_i \notin \mathbb{N}_0^l$ . This shows  $\psi(\tilde{\lambda}) = 0$ , so that  $\psi(\lambda) = \sum_{i=1}^l \varphi(\lambda)_i e_{\sigma(i)}$ , hence  $\varphi(\lambda)_i = \psi(\lambda)_{\sigma(i)}$ .  $\square$

**Definition 3.3.** Suppose  $\varphi$  is a function as in 3.1 for  $G$ . Letting  $e_1, \dots, e_l$  be the standard basis of  $\mathbb{Z}^l$  and  $d_1, \dots, d_l$  be the preimages of the  $e_i$  in  $X_0(T)$  with respect to  $\varphi$ , we set  $P_r(D) = \{\lambda \in P(D) \cap X_r(T) \mid \forall i \in \{1, \dots, l\} : \lambda - p^r d_i \notin P(D)\}$ .

It follows from the proof of 3.2 that  $P_r(D)$  does not depend on the choice of  $\varphi$  or the  $d_i$ .

**Lemma 3.4.** Let  $\lambda = \lambda_0 + p^r \tilde{\lambda} = \lambda'_0 + p^r \tilde{\lambda}'$  with  $\lambda_0, \lambda'_0 \in X_r(T)$ ,  $\tilde{\lambda}, \tilde{\lambda}' \in X(T)$ . Then there is  $\mu \in X_0(T)$  such that  $\lambda_0 = \lambda'_0 + p^r \mu$ ,  $\tilde{\lambda} = \tilde{\lambda}' + \mu$ .

*Proof.* We have  $\lambda_0 - \lambda'_0 = p^r \mu$ , where  $\mu = \tilde{\lambda} - \tilde{\lambda}'$ . For all  $\alpha \in R$ , we have  $p^r \langle \mu, \alpha^\vee \rangle = \langle \lambda_0 - \lambda'_0, \alpha^\vee \rangle = \langle \lambda_0, \alpha^\vee \rangle - \langle \lambda'_0, \alpha^\vee \rangle$ . As  $\lambda_0, \lambda'_0 \in X_r(T)$ , we have  $-(p^r - 1) \leq \langle \lambda_0, \alpha^\vee \rangle - \langle \lambda'_0, \alpha^\vee \rangle \leq p^r - 1$ , forcing  $\langle \mu, \alpha^\vee \rangle = 0$ . Thus,  $\mu \in X_0(T)$  and the result follows.  $\square$

**Theorem 3.5.** Suppose 3.1 holds for  $G$ . If  $\lambda \in X_r(T) + p^r X(T)$ , then there is a unique decomposition  $\lambda = \lambda_0 + p^r \tilde{\lambda}$  with  $\lambda_0 \in P_r(D)$ ,  $\tilde{\lambda} \in X(T)$ . Furthermore, if all weights of the simple  $G_r T$ -module  $\widehat{L}_r(\lambda)$  belong to  $P(D)$ , then  $\lambda \in P_r(D) + p^r P(D)$ .

*Proof.* Let  $\lambda = \lambda_0 + p^r \tilde{\lambda}$  with  $\lambda_0 \in X_r(T)$ ,  $\tilde{\lambda} \in X(T)$ . As  $d_i \in X_0(T)$ , we get that for each  $a \in \mathbb{Z}^l$ ,  $\lambda_0 + p^r \sum_{i=1}^l a_i d_i + p^r (\tilde{\lambda} - \sum_{i=1}^l a_i d_i)$  is another decomposition such that  $\lambda_0 + p^r \sum_{i=1}^l a_i d_i \in X_r(T)$ . Since  $d_1, \dots, d_l$  generate  $X_0(T)$ , 3.4 shows that every decomposition of  $\lambda$  arises in this fashion. By (1), (3), (4) and since  $W$  acts trivially on  $X_0(T)$ , there is a unique  $a \in \mathbb{Z}^l$  such that  $\lambda_0 + p^r \sum_{i=1}^l a_i d_i \in P_r(D)$ . Thus, there is a unique decomposition  $\lambda = \lambda_0 + p^r \tilde{\lambda}$  such that  $\lambda_0 \in P_r(D)$ . We show by contraposition that if all weights of  $\widehat{L}_r(\lambda)$  are polynomial, then  $\tilde{\lambda} \in P(D)$ . Suppose that  $\tilde{\lambda} \notin P(D)$ , so that  $\varphi(\tilde{\lambda}) \notin \mathbb{N}_0^l$ . Then there is  $i \in \{1, \dots, l\}$  such that  $\varphi(\tilde{\lambda})_i < 0$ . Since  $\lambda_0 - p^r d_i \notin P(D)$ , (1), (3), (4) yield  $\varphi(\lambda_0)_i < p^r$ . Using (2) and (3), we see that there is  $w \in W$  such that  $\varphi(w \cdot \lambda_0 + p^r \tilde{\lambda})_i = \varphi(\lambda_0)_i + p^r \varphi(\tilde{\lambda})_i < 0$ , so that

$w.\lambda_0 + p^r\tilde{\lambda} \notin P(D)$  by (1). Since  $\widehat{L}_r(\lambda_0)$  lifts to a simple  $G$ -module by [7, II.9.6.(g)], its character is  $W$ -invariant, so that  $w.\lambda_0$  is a weight of  $\widehat{L}_r(\lambda_0)$  and  $w.\lambda_0 + p^r\tilde{\lambda}$  is a weight of  $\widehat{L}_r(\lambda_0 + p^r\tilde{\lambda}) \cong \widehat{L}_r(\lambda_0) \otimes_k p^r\tilde{\lambda}$ , so that not all weights of  $\widehat{L}_r(\lambda)$  are polynomial.  $\square$

For  $G = \mathrm{GL}_n$ , it was shown in the Appendix of [9] that a  $G_rT$ -module such that all of its weights are polynomial is a polynomial  $G_rT$ -module. It is not known for which general  $G$  a similar statement holds. However, if  $M$  is normal as a variety, we can at least prove this for some simple modules.

**Proposition 3.6.** *Suppose that  $M$  is a normal variety. If  $\lambda = \lambda_0 + p^r\tilde{\lambda} \in (X_r(T) \cap P(D)) + p^rP(D)$ , the simple  $G_rT$ -module  $\widehat{L}_r(\lambda)$  is a polynomial  $G_rT$ -module.*

*Proof.* We have  $\widehat{L}_r(\lambda) \cong \widehat{L}_r(\lambda_0) \otimes_k p^r\tilde{\lambda}$  and  $\widehat{L}_r(\lambda_0)$  lifts to  $G$  by [7, II.9.6]. Since  $M$  is normal and  $\lambda_0$  is a dominant weight contained in  $P(D)$ ,  $\widehat{L}_r(\lambda_0)$  lifts to  $M$  by [2, 3.5], so that  $\widehat{L}_r(\lambda_0)$  is a polynomial  $G_rT$ -module. As  $\tilde{\lambda}$  is a  $D$ -module,  $p^r\tilde{\lambda}$  is an  $F^{-r}(D)$ -module, where  $F^r$  is the  $r$ -th iteration of the Frobenius morphism, and thus a polynomial  $G_rT$ -module by restriction. Hence  $\widehat{L}_r(\lambda)$  is a polynomial  $G_rT$ -module.  $\square$

**Corollary 3.7.** *Let  $M$  be normal and suppose that 3.1 holds for  $G$ . If  $\lambda \in X_r(T) + p^rX(T)$ , then the simple module  $\widehat{L}_r(\lambda)$  is a polynomial  $G_rT$ -module iff  $\lambda \in P_r(D) + p^rP(D)$ .*

*Proof.* This follows directly from 3.5 and 3.6.  $\square$

**Corollary 3.8.** *Let the derived subgroup of  $G$  be simply connected and  $M$  be normal. Suppose 3.1 holds for  $G$ . Then a system of representatives of isomorphism classes of simple polynomial  $G_rT$ -modules is given by  $\{\widehat{L}_r(\lambda) | \lambda \in P_r(D) + p^rP(D)\}$ .*

*Proof.* Let  $\lambda \in X(T)$ . Since the derived subgroup of  $G$  is simply connected,  $X_r(T)$  contains a set of representatives of  $X(T)/p^rX(T)$ , so that we have  $\lambda \in X_r(T) + p^rX(T)$ . By 3.5, all weights of  $\widehat{L}_r(\lambda)$  are polynomial iff  $\lambda \in P_r(D) + p^rP(D)$  and by 3.6,  $\widehat{L}_r(\lambda)$  lifts to  $G_rT$  in this case. Since all weights of polynomial  $G_rT$ -modules are polynomial and a polynomial module is a simple polynomial  $G_rT$ -module iff it is simple as a  $G_rT$ -module, the result follows.  $\square$

We now give an explicit construction for a function as in 3.1 under certain assumptions.

**Theorem 3.9.** *Suppose there is an action of  $W$  on  $X(T')$  such that the canonical projection  $\pi : X(T') \rightarrow X(T)$  is  $W$ -equivariant and there are  $b_1, \dots, b_s \in \pi^{-1}(X_0(T)) \cap P(D')$  such that*

(a)  $\forall i \in \{1, \dots, s\}, j \in \{1, \dots, n\} : (b_i)_j \in \{0, 1\}$ ,

- (b) the sets  $M_i = \{j \in \{1, \dots, n\} \mid (b_i)_j = 1\}$  form a partition of  $\{1, \dots, n\}$ ,
- (c) the image of  $W$  in  $S_{X(T')}$  is contained in the image of  $W' \cong S_n$  and contains the subgroup  $S_{M_1} \times \dots \times S_{M_s}$ ,
- (d)  $\exists d_1, \dots, d_l \in \{b_1, \dots, b_s\} : \pi(d_1), \dots, \pi(d_l)$  is a basis of  $X_0(T)$  such that each  $\pi(b_i)$  is a linear combination of the  $\pi(d_i)$  with non-negative coefficients.

Then there is a function  $\varphi : X(T) \rightarrow \mathbb{Z}^l$  with the properties of 3.1.

*Proof.* For every  $i \in \{1, \dots, s\}$ , write  $\pi(b_i) = \sum_{j=1}^l n_{ij} \pi(d_j)$  with  $n_{ij} \in \mathbb{N}_0$ . Define  $\varphi : X(T') \rightarrow \mathbb{Z}^l$ ,  $\lambda \mapsto \sum_{j=1}^l \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} n_{ij} e_j$ . We show that  $\varphi$  induces a function  $\bar{\varphi} : X(T) \rightarrow \mathbb{Z}^l$  with the properties of 3.1. For this, we show the following statements (i) – (v).

- (i)  $\forall \lambda \in X(T'), \mu \in \ker(\pi) : \varphi(\lambda + \mu) = \varphi(\lambda) :$

Let  $\mu \in \ker(\pi)$ . Since the sets  $M_i$  are pairwise disjoint, we have  $\mu' = \mu - \sum_{i=1}^s \min\{\mu_a \mid a \in M_i\} b_i \in P(D')$  and for every  $i$ , there is  $a \in M_i$  such that  $\mu'_a = 0$ . By 3.10 below and (a), we get  $\mu'_l = 0$  for all  $l \in M_i$ . As the  $M_i$  form a partition of  $\{1, \dots, n\}$ , we get  $\mu' = 0$ , so that  $\mu = \sum_{i=1}^s \min\{\mu_a \mid a \in M_i\} b_i$ . Consequently,  $\varphi(\lambda + \mu) = \varphi(\lambda) + \varphi(\mu)$  for all  $\lambda \in X(T')$ . As  $\mu \in \ker(\pi)$ , we have

$$\begin{aligned} 0 = \pi(\mu) &= \sum_{i=1}^s \min\{\mu_a \mid a \in M_i\} \pi(b_i) \\ &= \sum_{j=1}^l \left( \sum_{i=1}^s n_{ij} \min\{\mu_a \mid a \in M_i\} \right) \pi(d_j), \end{aligned}$$

so that  $\sum_{i=1}^s n_{ij} \min\{\mu_a \mid a \in M_i\} = 0$  for every  $j$  and  $\varphi(\mu) = 0$ . Hence (i) holds.

- (ii)  $\forall \lambda \in X(T') : \varphi(p^r \lambda) = p^r \varphi(\lambda) :$

Clear by definition.

- (iii)  $\forall \lambda \in X(T') : \varphi(\lambda) \in \mathbb{N}_0^l \iff \lambda \in P(D) + \ker(\pi), :$

Suppose  $\varphi(\lambda) \in \mathbb{N}_0^l$ . We have  $\lambda - \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} b_i \in P(D')$ , so that  $\pi(\lambda) - \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} \pi(b_i) \in \pi(P(D'))$ . Hence  $\pi(\lambda) \in \pi(P(D')) + \sum_{j=1}^l \sum_{i=1}^s \min\{\lambda_a \mid a \in M_i\} n_{ij} \pi(d_j)$ . By assumption, the coefficients in the linear combination are non-negative, so that this set is contained in  $\pi(P(D'))$  and  $\lambda \in P(D') + \ker(\pi)$ . Now let  $\lambda = \lambda' + \mu \in P(D') + \ker(\pi)$ . Since  $\varphi(\lambda) = \varphi(\lambda')$  by (i) and since  $\lambda'$  has no negative coordinates and  $n_{ij} > 0$ , we have  $\varphi(\lambda) \in \mathbb{N}_0^l$ .

- (iv)  $\forall \lambda, \lambda' \in X(T') : \exists w \in W : \varphi(w \cdot \lambda + \lambda') = \varphi(\lambda) + \varphi(\lambda') :$

Using (c), we permute the coordinates in each  $M_i$  for  $\lambda$  in such a way that the lowest coordinate for  $w \cdot \lambda$  and  $\lambda'$  in each  $M_i$  occurs at the same place. Then clearly  $\varphi(w \cdot \lambda + \lambda') = \varphi(\lambda) + \varphi(\lambda')$ .

(v)  $\varphi(d_i) = e_i$ :

This follows directly from (d) and the definition of  $\varphi$ .

The statement (i) implies that there is a function  $\bar{\varphi} : X(T) \rightarrow \mathbb{Z}^l$  such that  $\varphi = \bar{\varphi} \circ \pi$ . Now (ii) – (v) imply that  $\bar{\varphi}$  has the properties of 3.1.

□

**Lemma 3.10.** *In the situation of 3.9, let  $i \in \{1, \dots, s\}$  and  $\mu \in \ker(\pi)$ . Then  $\mu_k = \mu_l$  for all  $k, l \in M_i$ .*

*Proof.* Let  $\mu \in \ker(\pi)$  and  $i \in \{1, \dots, s\}$ . Suppose there are  $k, l \in M_i$  such that  $\mu_k \neq \mu_l$ . Let  $w = (k \ l) \in W$  be the transposition with respect to  $k$  and  $l$ . Then  $\mu - w.\mu = (\mu_k - \mu_l)(e_k - e_l) \in \ker(\pi)$ . As the restriction of this element is zero in  $X(T)$  and  $X(T)$  is torsion-free,  $(e_k - e_l)|_T = 0$ , so that  $e_k|_T = e_l|_T$ . It follows that  $w$  acts trivially on  $T$ , a contradiction. □

The function  $\varphi$  defined in the proof of 3.9 provides an easy way to check whether the restriction of a character  $\lambda \in X(T')$  to  $T$  is a polynomial weight.

We now apply our results to several examples. The first part of the following corollary is [5, Corollary 3.2]. For the orthogonal and symplectic similitude groups  $GO_{2l+1}$  and  $GSp_{2l}$ , we adopt the definition and notation from [1, Sections, 5, 7]. Recall that with suitable choice of defining form, the maximal torus of diagonal matrices in  $GSp_{2l}$  resp.  $GO_{2l+1}$  is given by  $\{\text{diag}(t_1, \dots, t_{2l}) | t_{ii}t_{i'i'} = t_{jj}t_{j'j'} \text{ for } 1 \leq i, j \leq l\}$  resp.  $\{\text{diag}(t_1, \dots, t_{2l+1}) | t_{ii}t_{i'i'} = t_{jj}t_{j'j'} \text{ for } 1 \leq i, j \leq l+1\}$ .

**Corollary 3.11.** (a) *For  $G = \text{GL}_n, T$  the torus of diagonal matrices, a complete set of representatives of simple polynomial  $G_r T$ -modules is given by  $\{\widehat{L}_r(\lambda) | \lambda \in P_r(D) + p^r P(D)\}$ , where  $P_r(D) = \{\lambda \in P(D) | 0 \leq \lambda_i - \lambda_{i+1} \leq p^r - 1 \text{ for } 1 \leq i \leq n-1, 0 \leq \lambda_n \leq p^r - 1\}$ .*  
 (b) *Let  $G = GSp_{2l}, T$  the torus of diagonal matrices in  $G$ . Then a complete set of representatives of the isomorphism classes of simple polynomial  $G_r T$ -modules is given by  $\{\widehat{L}_r(\lambda) | \lambda \in P_r(D) + p^r P(D)\}$ , where  $P_r(D) = \{\lambda \in P(D) \cap X_r(T) | \lambda - p^r d \notin P(D)\}$  and  $d$  is the character of  $T$  given by  $t \mapsto t_{11}t_{nn}$ .*  
 (c) *Let  $G = GO_{2l+1}, T$  the torus of diagonal matrices in  $G$ . If  $\lambda \in X_r(T) + p^r X(T)$  and all weights of the module  $\widehat{L}_r(\lambda)$  are polynomial, then  $\lambda \in P_r(D) + p^r P(D)$ , where  $P_r(D) = \{\lambda \in P(D) \cap X_r(T) | \lambda - p^r d \notin P(D)\}$  and  $d$  is the character of  $T$  given by  $t \mapsto t_{l+1}$ .*  
 (d) *Let  $n_i \in \mathbb{N}$  such that  $n = \sum_{i=1}^l n_i$  and embed  $G = \text{GL}_{n_1} \times \dots \times \text{GL}_{n_l}$  into  $\text{GL}_n$  as a Levi subgroup in the obvious way. Let  $M_i = \{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$  and  $d_i = \sum_{j \in M_i} e_j$  for  $1 \leq i \leq l$ . Then a complete set of representatives of isomorphism classes of simple polynomial  $G_r T$ -modules is given by  $\{\widehat{L}_r(\lambda) | \lambda \in P_r(D) +$*



$p^r P(D)\}$ , where  $P_r(D) = \{\lambda \in X_r(T) \cap P(D) \mid \lambda - p^r d_i \notin P(D) \text{ for } 1 \leq i \leq l\}$ .

- Proof.* (a) Let  $d = \det|_T = (1, \dots, 1)$ ,  $M = \{1, \dots, n\}$ . Clearly  $d$  and  $M$  have the properties of 3.9 and  $P_r(D)$  has the required form. As the derived subgroup of  $\mathrm{GL}_n$  is simply connected and  $\mathrm{Mat}_n$  is normal, the result follows from 3.8.
- (b) We have  $W \cong S_l \ltimes (\mathbb{Z}/(2))^l$ . We can identify  $W$  with a subgroup of  $W' \cong S_{2l}$  by letting  $\sigma \in S_l$  act on  $\lambda \in X(T')$  via  $(\sigma.\lambda)_i = \lambda_{\sigma^{-1}(i)}$ ,  $(\sigma.\lambda)_{i'} = \lambda_{(\sigma^{-1}(i))'}$  for  $1 \leq i \leq l$  and letting the  $i$ -th unit vector of  $(\mathbb{Z}/(2))^l$  act as the transposition  $(ii')$ . With this action, the canonical projection  $\pi : X(T') \rightarrow X(T)$  is  $W$ -equivariant. For  $1 \leq i \leq l$ , let  $d_i = e_i + e_{i'}$  and  $M_i = \{i, i'\}$ . Then  $\pi(d_i) = \pi(d_j)$  for  $1 \leq i, j \leq l$ , the character  $d = \pi(d_1)$  is a basis of  $X_0(T)$  and the  $M_i$  form a partition of  $\{1, \dots, n\}$ . Inside  $W \cong S_l \ltimes (\mathbb{Z}/(2))^l$ ,  $(\mathbb{Z}/(2))^l$  corresponds to  $S_{M_1} \times \dots \times S_{M_l}$ . Thus, 3.9 yields that  $G$  has the properties of 3.1 and  $P_r(D)$  has the desired form. The derived subgroup of  $G$  is  $Sp_{2l}$ , hence simply connected, and by the remark below 2.11 in [3],  $\overline{G}$  is normal. The result now follows from 3.8.
- (c) For  $1 \leq i \leq l$ , let  $d_i = e_i + e_{i'}$ ,  $M_i = \{i, i'\}$  and let  $d_{l+1} = e_{l+1}$ ,  $M_{l+1} = \{l+1\}$ . Then the  $M_i$  form a partition of  $\{1, \dots, l\}$ , the character  $d = \pi(d_{l+1})$  is a basis of  $X_0(T)$  and  $\pi(d_i) = 2\pi(d)$  for  $1 \leq i \leq l$ . The other properties of 3.9 can be checked as in the proof of (b), so that  $G$  has the properties of 3.1 and  $P_r(D)$  has the desired form. The result now follows from 3.5.
- (d) The  $M_i$  form a partition of  $\{1, \dots, n\}$ . By definition of  $G$ , we have  $W = S_{M_1} \times \dots \times S_{M_l} \subseteq W' = S_n$  and the  $d_i$  form a basis of  $X_0(T)$ . Thus, 3.9 shows that  $G$  has the properties of 3.1, so that the result follows from 3.8.  $\square$

We give an example showing that our approach does not work for the connected component  $G$  of the even orthogonal similitude group  $GO_{2l}$ . Let  $l = 4$  and choose  $p, r$  such that  $4 \mid p^r - 1$ . Let  $T, T', d$  be as in 3.11.(b). Let  $\varphi : X(T') \rightarrow \mathbb{Z}$ ,  $t \mapsto \sum_{i=1}^l \min\{t_i, t_{i'}\}$ . The same proof shows that with the exception of (iv), all statements in the proof of 3.9 hold for  $\varphi$  and  $d$ . Let  $\lambda_0, \tilde{\lambda} \in X(T')$  be the characters given by  $\lambda_0 = (\frac{p^r-1}{2}, \frac{p^r-1}{2}, \frac{p^r-1}{2}, \frac{p^r-1}{2}, \frac{p^r-1}{4}, \frac{p^r-1}{4}, \frac{p^r-1}{4}, \frac{p^r-1}{4})$  and  $\tilde{\lambda} = (0, 0, 0, 0, 1, 1, -1, 1)$ . Then  $\varphi(\lambda_0) = p^r - 1 > 0$ ,  $\varphi(\lambda_0 - p^r d) = -1 < 0$  and  $\lambda_0|_T \in X_r(T)$ , so that  $\lambda_0|_T \in P_r(D)$ . In  $W \cong S_l \ltimes (\mathbb{Z}/(2))^{l-1}$ ,  $S_l$  acts on  $X(T')$  as in the proof of 3.11.(b) while  $(\mathbb{Z}/(2))^{l-1}$  acts as the subgroup  $\{x \in (\mathbb{Z}/(2))^l \mid x_i \neq 0 \text{ for an even number of } i\}$  of  $(\mathbb{Z}/(2))^l$ . Thus,  $\lambda_0$  is invariant under  $S_l$  and we have  $\varphi(w.\lambda_0|_T + p^r \tilde{\lambda}|_T) > 0$  and hence  $w.\lambda_0|_T + p^r \tilde{\lambda}|_T \in X(D)$  for all  $w \in (\mathbb{Z}/(2))^{l-1}$ . Hence the arguments in the proof of 3.5 are not applicable in this case. As the statement  $w.\lambda_0|_T + p^r \tilde{\lambda}|_T \in X(D)$  does not depend on  $\varphi$  and there

is no other choice for  $d$  in this case, we cannot find other  $\varphi$  or  $d$  in this example to salvage the proof of 3.5. However, since we have not computed the weights of  $\widehat{L_r}(\lambda_0|_T + p^r \tilde{\lambda}|_T)$ , we do not know whether all weights of this module are in fact polynomial.

#### 4. EQUIVALENCES BETWEEN BLOCKS OF POLYNOMIAL REPRESENTATIONS

We adopt the notation of the previous section. Suppose that  $G$  has simply connected derived subgroup and that there is  $0 \neq b \in X_0(T) \cap P(D)$  such that  $\langle b \rangle = X_0(T)$ . Choose representatives  $(\omega_\alpha)_{\alpha \in \Delta}$  of the fundamental dominant weights of the derived subgroup such that for all  $\alpha \in \Delta$ ,  $\omega_\alpha \in P(D)$  and  $\omega_\alpha - b \notin P(D)$ . Then the  $\omega_\alpha$  together with  $b$  form a basis of  $X(T)$ . Let  $\varphi : X(T) \rightarrow \mathbb{Z}$  be the projection on the coordinate corresponding to  $b$  and define  $P_r(D) = \{\lambda \in X_r(T) \cap P(D) \mid 0 \leq \varphi(\lambda) \leq p^r - 1\}$ . Then each  $\lambda \in X(T)$  can be written uniquely as  $\lambda = \lambda_0 + p^r \tilde{\lambda}$  with  $\lambda_0 \in P_r(D)$ ,  $\tilde{\lambda} \in X(T)$ . By [7, II.1.18], we have  $X(G) \cong X_0(T)$  via restriction, so that we can regard  $b$  as a character of  $G$  and have a shift functor  $[b] : \text{mod } G_r T \rightarrow \text{mod } G_r T$ ,  $V \mapsto V[b] = V \otimes_k k_b$ , where  $k_b$  is the one-dimensional  $G$ -module induced by  $b$ . This shift functor is an autoequivalence of  $\text{mod } G_r T$  with inverse  $[-b]$ , hence it induces equivalences between blocks of  $\text{mod } G_r T$ . Since  $-b$  is not a polynomial weight,  $[-b]$  does not restrict to an autoequivalence of the category of polynomial  $G_r T$ -modules. However, we will see in this section that the functor  $[b]$  sometimes still induces an equivalence between blocks of this category. Let  $W_p \cong W \ltimes p\mathbb{Z}R$  be the affine Weyl group of  $G$ , let  $R^+$  be a set of positive roots and  $\rho = \sum_{\alpha \in R^+} \frac{1}{2}\alpha$  be the half-sum of positive roots. We denote by  $w \bullet \lambda = w(\lambda + \rho) - \rho$  the dot-action of  $w \in W_p$  on  $\lambda \in X(T)$ . For  $a \in \mathbb{Z}$ , if  $a = qp + r$  for  $q, r \in \mathbb{Z}$  such that  $0 \leq r \leq p - 1$ , we write  $r = a \pmod p$ .

**Lemma 4.1.** *Let the derived subgroup of  $G$  be simply connected. Let  $\lambda \in P_r(D) + p^r P(D)$  and  $a = \max_{w \in W} \{\varphi(w \cdot \lambda + w \cdot \rho - \rho) \pmod p\}$ . Then for  $1 \leq i \leq p - a - 1$ , the shift  $[ib]$  defines a bijection*

$$W_p \bullet \lambda \cap (P_r(D) + p^r P(D)) \rightarrow W_p \bullet (\lambda + ib) \cap (P_r(D) + p^r P(D)).$$

*Proof.* Let  $(w, pl\mu) \in W_p$ . We show that  $(w, pl\mu) \bullet \lambda \in P_r(D) + p^r P(D)$  iff  $(w, pl\mu) \bullet (\lambda + ib) \in P_r(D) + p^r P(D)$ . We have  $(w, pl\mu) \bullet \lambda = w \cdot \lambda + w \cdot \rho - \rho + pl\mu$  and  $(w, pl\mu) \bullet (\lambda + ib) = (w, pl\mu) \bullet \lambda + ib$  by definition of the dot-action and since  $w \cdot b = b$ . Writing  $w \cdot \lambda + w \cdot \rho - \rho + pl\mu = \nu_0 + p^r \tilde{\nu}$  with  $\nu_0 \in P_r(D)$  and  $\tilde{\nu} \in X(T)$ , we have  $\varphi(\nu_0) \pmod p = \varphi(w \cdot \lambda + w \cdot \rho - \rho) \pmod p \leq a$ . It follows that  $(\varphi(\nu_0) \pmod p) + i \leq a + p - a - 1 = p - 1$ . Thus,  $\varphi(\nu_0 + ib) \leq p^r$ , so that  $\nu_0 + ib \in P_r(D)$ . By unicity of decomposition, we get  $(w, pl\mu) \bullet \lambda \in P_r(D) + p^r P(D) \iff \tilde{\nu} \in P(D) \iff (w, pl\mu) \bullet (\lambda + ib) \in P_r(D) + p^r P(D)$ .  $\square$

For  $G = \mathrm{GL}_n$ , we can choose  $b = \det$  and have  $a = \max_{w \in W} \{(w.\lambda)_n + (w.\rho)_n - \rho_n \bmod p\}$ . We also have that every  $G_rT$ -module  $V$  such that all weights of  $V$  are polynomial lifts to  $\overline{G_rT} = M_rD$  by Jantzen's result in [9, 5.4], implying  $\mathrm{Ext}_{G_rT}^1(V, W) \cong \mathrm{Ext}_{M_rD}^1(V, W)$  for all  $V, W \in \mathrm{mod} M_rD$ . For large  $r$ , the infinitesimal Schur algebra  $S_d(G_rT)$  is isomorphic to  $S_d(G)$ , so that the following proposition can also be applied to blocks of the ordinary Schur algebra of  $\mathrm{GL}_n$ .

**Proposition 4.2.** *Let  $G = \mathrm{GL}_n$ ,  $T$  the torus of diagonal matrices. Let  $\lambda \in P_r(D) + p^r P(D)$  and  $a = \max_{w \in W} \{(w.\lambda)_n + (w.\rho)_n - \rho_n \bmod p\}$ . Then for  $1 \leq i \leq p - a - 1$ , the functor  $[i \det]$  defines an equivalence between the blocks of  $\mathrm{mod} M_rD$  containing  $\widehat{L}_r(\lambda)$  and  $\widehat{L}_r(\lambda + i \det)$ .*

*Proof.* Let  $B$  be the block containing  $\lambda$  and  $B'$  be the block containing  $\lambda + i \det$ . As  $[i \det]$  maps nonsplit extensions in  $\mathrm{mod} M_rD$  to nonsplit extensions in  $\mathrm{mod} M_rD$ , we have  $B[i \det] \subseteq B'$ . Let  $\mu \in B'$ . Suppose there is  $\nu \in B[i \det]$  such that  $\mathrm{Ext}_{M_rD}^1(\widehat{L}_r(\mu), \widehat{L}_r(\nu)) \neq 0$ . Then  $\mu \in W_p \bullet (\lambda + i \det)$  since  $B' \subseteq W_p \bullet (\lambda + i \det)$  by [7, II.9.19]. Now 4.1 yields  $\mu - i \det \in P_r(D) + p^r P(D)$ . Applying  $[-i \det]$  to a nonzero element of  $\mathrm{Ext}_{M_rD}^1(\widehat{L}_r(\mu), \widehat{L}_r(\nu))$ , we get  $\mathrm{Ext}_{G_rT}^1(\widehat{L}_r(\mu - i \det), \widehat{L}_r(\nu - i \det)) \neq 0$ . Since both simple modules lift to  $M_rD$ , Jantzen's result yields  $\mathrm{Ext}_{G_rT}^1(\widehat{L}_r(\mu - i \det), \widehat{L}_r(\nu - i \det)) \cong \mathrm{Ext}_{M_rD}^1(\widehat{L}_r(\mu - i \det), \widehat{L}_r(\nu - i \det)) \neq 0$ , so that  $\mu - i \det \in B$  and  $\mu \in B[i \det]$ .  $\square$

We do not know whether an analogue of [9, 5.4] and hence a generalization of the foregoing result holds for other reductive group schemes.

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